9. For $c \in \mathbb{Q}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with $x_n \to c$. Since $f(x_n) \to 0 < f(c)$, f is not continuous at c.
   For $c \in \mathbb{R} \setminus \mathbb{Q}$, let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ where p_n and q_n are in $\mathbb{N}$, each $\frac{p_n}{q_n}$ is in lowest terms, and $\frac{p_n}{q_n} \to c$. As in Example 4.8, $q_n \to \infty$. Thus $f\left(\frac{p_n}{q_n}\right) = q_n \to \infty \neq 0 = f(c)$, and so f is not continuous at c.

4.3 Limits of Functions

1. Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and let $g = -f$. Then neither f nor g has a limit at 0. Since $f + g$ is the constant function 0, and fg is the constant function -1, both $f + g$ and fg have limits at 0.

2. For $c \in \mathbb{R}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q} \setminus \{c\}$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \setminus \{c\}$ with $x_n \to c$ and $y_n \to c$. Then $f(x_n) \to 1$ and $f(y_n) \to -1$. By Proposition 4.6, $\lim_{x\to c} f(x)$ does not exist.

3. As in Exercise 4.2.5 for $f(x) = \cos\frac{1}{x}$, let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ $\forall\, n \in \mathbb{N}$. Then $x_n \to 0$ and $f(x_n) \to 1$, while $y_n \to 0$ and $f(y_n) \to 0$. By Proposition 4.6, $\lim_{x\to 0} f(x)$ does not exist.
   Next let $\varepsilon > 0$, let $\delta = \varepsilon$, and let $0 < |x - 0| < \delta$. Then $\left|x \cos\frac{1}{x} - 0\right| \leq |x| < \delta = \varepsilon$, and so $\lim_{x\to 0} x\cos\frac{1}{x} = 0$. By the Remark following Example 4.14, $g(x) = x\cos\frac{1}{x}$ for $x \neq 0$ can be continuously extended to 0 by defining $g(0) = 0$.

4. Let $L = \lim_{x\to c} f(x)$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D \setminus \{c\}$ such that $x_n \to c$. By Proposition 4.6 and the paragraph before Example 4.15, $f(x_n) \to L$. Since $a \leq f(x_n) \leq b\ \forall\, n \in \mathbb{N}$, Theorem 3.3 implies that $a \leq L \leq b$. Alternatively, one could prove this directly from Definition 4.3. If $L < a$, let V be a neighborhood of L lying entirely to the left of a. By Definition 4.3, $\exists$ a neighborhood U of c such that $x \in (U \cap D) \setminus \{c\} \Rightarrow f(x) \in V$. Hence, $x \in (U \cap D) \setminus \{c\} \Rightarrow f(x) < a$,

a contradiction. Therefore, $L \geq a$. Similarly, if $L > b$, let V be a neighborhood of L lying entirely to the right of b. As above, $\exists$ a neighborhood U of c such that $x \in (U \cap D) \setminus \{c\} \Rightarrow f(x) \in V$ and so $f(x) > b$, a contradiction. Therefore, $L \leq b$.

5. Using neighborhoods we can do all cases (c real or $c = \pm\infty$) at once. Let V be a neighborhood of L. By Definitions 4.3 and 4.4, $\exists$ neighborhoods U_1 of c and U_2 of c such that $x \in (U_1 \cap D) \setminus \{c\} \Rightarrow f(x) \in V$ and $x \in (U_2 \cap D) \setminus \{c\} \Rightarrow h(x) \in V$. Then $U = U_1 \cap U_2$ is a neighborhood of c and $x \in (U \cap D) \setminus \{c\} \Rightarrow$ both $f(x) \in V$ and $h(x) \in V$. Since $f(x) \leq g(x) \leq h(x)$, $x \in (U \cap D) \setminus \{c\} \Rightarrow g(x) \in V$. Therefore, $\lim_{x \to c} g(x) = L$. Alternatively, this follows from the Squeeze Theorem for sequences (Theorem 3.4) by using Proposition 4.6 for c real, and by using the analogous result to Proposition 4.6 for $c = \pm\infty$ (as mentioned in the paragraph before Example 4.15).

6. Let $L = \lim_{x \to c} f(x) \in \mathbb{R}$ and let $V = (L-1, L+1)$. Then $\exists$ a neighborhood U of c (by Definition 4.3 or 4.4) such that $x \in (U \cap D) \setminus \{c\} \Rightarrow f(x) \in V$, or equivalently, $|f(x) - L| < 1$. Hence, $x \in (U \cap D) \setminus \{c\} \Rightarrow |f(x)| - |L| \leq |f(x) - L| < 1$ and so $|f(x)| < 1 + |L|$. If $c \notin D$, then f is bounded on $U \cap D$ by $1 + |L|$. If $c \in D$, then f is bounded on $U \cap D$ by $\max\{1 + |L|, |f(c)|\}$.

7. Let $L = \lim_{x \to c} f(x) > 0$ and let $V = \left(\frac{1}{2}L, \frac{3}{2}L\right)$. Then $\exists$ a neighborhood U of c such that $x \in (U \cap D) \setminus \{c\} \Rightarrow f(x) \in V$, and so $f(x) > \frac{1}{2}L > 0$.

8. Let $L = \lim_{x \to c} f(x) \neq 0$ and let $\varepsilon = \frac{|L|}{2}$. Then $\exists$ a neighborhood U of c such that $x \in (U \cap D) \setminus \{c\} \Rightarrow |L| - |f(x)| \leq |L - f(x)| < \frac{|L|}{2}$, and so $|f(x)| > \frac{|L|}{2}$.

9. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $D \setminus \{c\}$ with $x_n \to c$. By Exercise 3.7.2, $f(x_n) \to \infty \Leftrightarrow \frac{1}{f(x_n)} \to 0$. By Proposition 4.6 and the analogous result when $c = \pm\infty$, $\lim_{x \to c} f(x) = \infty \Leftrightarrow \lim_{x \to c} \frac{1}{f(x)} = 0$.

10. Note. For a noncontinuous additive function on $\mathbb{R}$, see Hewitt and